Mathematics Formulae

1 Proportion

1. If $a : b = c : d$ then $ad = bc$.

2. Altertendo: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{c} = \frac{b}{d}$ or $a : c = b : d$.

3. Componendo: If $\frac{a}{b} = \frac{c}{d}$ then $(a + b) : b = (c + d) : d$ or $\frac{a+b}{b} = \frac{c+d}{d}$.

4. Dividento: If $\frac{a}{b} = \frac{c}{d}$ then $(a - b) : b = (c - d) : d$ or $\frac{a-b}{b} = \frac{c-d}{d}$.

5. Componendo-Dividento: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

6. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k$ then $k = \frac{la+mc+ne}{lb+md+nf}$ or $k = \sqrt{\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}}$ or $k = \sqrt{\frac{la^2+mc^2+ne^2}{lb^2+md^2+nf^2}}$ or $\sqrt[3]{\frac{la^3+mc^3+ne^3}{lb^3+md^3+nf^3}}$.

2 Progressions

7. $a, a+d, a+2d, \cdots, a+(n-1)d, \cdots$ is called an Arithmetic Progression($A.P.$) with common difference d.

8. The n^{th}-term of an $A.P$, is $t_n = a+(n-1)d$, where $n = 1, 2, 3, \cdots$.

9. The sum of first n-terms of an $A.P$ is $S_n = \frac{n}{2}(a + l) = \frac{n}{2}[2a + (n - 1)d]$.

10. If a, b, c are three consecutive terms of an $A.P$, then b is called the Arithmetic Mean $(A.M)$ of a and c. Thus $b = \frac{a+c}{2}$.

11. $a, ar, ar^2, \cdots, ar^{n-1}, \cdots$ is called a Geometric Progression $(G.P)$ with common ratio r.

12. The n^{th} term of $G.P$ is $t_n = ar^{n-1}$.

13. The sum of first n-terms of a $G.P$ is $S_n = \frac{a(1-r^n)}{1-r}$ if $|r| < 1$ and $S_n = \frac{a(r^n-1)}{r-1}$ if $|r| > 1$

14. If $|r| < 1$, then sum of infinity number of terms of $G.P$ is $S_\infty = \frac{a}{1-r}$.

15. If a, b, c are three consecutive terms of a $G.P$ then b is called the Geometric Mean $(G.M)$ of a and c. Thus $b = \sqrt{ac}$.

16. $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \cdots, \frac{1}{a+(n-1)d}, \cdots$ is called a Harmonic Progression $(H.P)$.

17. The n^{th}-term of a $H.P$ is $t_n = \frac{1}{a+(n-1)d}$.

18. The Harmonic mean of a and b is n if a, n, b are in $H.P.$. Thus $n = \frac{2ab}{a+b}$.

3 Determinants

19. If all the elements of a row (or column) are zero, then the value of the determinant is zero.

20. The value of the determinant remains unchanged if its corresponding rows and columns are interchanged, that is $|A| = |A^T|$.

21. If any two rows (or columns) are interchanged, then the value of the determinant is multiplied by (-1).

22. If two rows (or columns) are identical, the the value of the determinant is zero.

23. If the corresponding elements of two rows (or columns) are proportional to each other, then the value of the determinant is zero.

24. The value of the determinant of a diagonal matrix or a lower triangular matrix or an upper triangular matrix is the product of its diagonal elements.

25. If each element of a row (or column) is multiplied by a scalar α, then the value of the determinant is multiplied by the scalar α.

26. If β is a factor of each element of a row (or column), then this factor β can be taken out of the determinant.

27. If A is a square matrix of order n and α is a scalar, then $|\alpha A| = \alpha^n |A|$.

28. If each element of any row (or column) can be written as the sum of two (or more) terms, then the determinant can be written as sum of two (or more) determinants.

29. In a non-zero multiple of the elements of some row (or column) is added to the corresponding elements of some other row (or column), then the value of the determinant remains unchanged.

30. If A and B are two square matrices of same order, then $|AB| = |A||B|$.

4 Matrix Algebra

31. $A + B = B + A$

32. $A + (B + C) = (A + B) + C$

33. $AB \neq BA$

34. $AB = AC$ does not necessarily imply $B = C$

35. $AB = 0$ does not necessarily imply $A = 0, B = 0$, or $BA = 0$

36. $A(BC) = (AB)C$

37. $AI = IA = A$

38. A square matrix A is said to be symmetric if $A^T = A$

39. A square matrix A is said to be skew-symmetric if $A^T = -A$

40. A square matrix A is said to be orthogonal if $AA^T = A^T A = I$

41. A square complex matrix A is said to be Hermitian if $\overline{A}^T = A$

42. A square complex matrix A is said to be skew-Hermitian if $\overline{A}^T = -A$

43. A square complex matrix A is said to be Unitary if $A\overline{A}^T = \overline{A}^T A = I$

44. Let A be a square matrix, λ be a scalar. If there exist a non-zero column vector X such that $AX = \lambda X$, then λ is the eigen value of A and X be the corresponding eigen vector.

45. The characteristic equation of A is $|A - \lambda I| = 0$

46. Calyley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

47. If λ is an eigen value of A, then λ^m is an eigen value of A^m.

48. If λ is an eigen value of A, then $\lambda + k$ is an eigen value of $A + kI$, where k is a scalar.

49. If $\lambda \neq 0$ is an eigen value of A, then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

5 Algebra

50. $(a + b)^2 = a^2 + 2ab + b^2$

51. $(a - b)^2 = a^2 - 2ab + b^2$

52. $(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$

53. $(a + b)(a - b) = a^2 - b^2$

54. $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

55. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

56. $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

57. $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

58. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1})$ where n is a positive integer.

59. $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1})$ where n is an odd positive integer.

60. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

61. If $a+b+c = 0$ then, $a^3+b^3+c^3-3abc = 0$.

62. ${}_nP_r = n(n - 1)(n - 2)\cdots(n - \overline{r - 1})$

63. ${}_nP_r = \frac{n!}{(n-r)!}$

64. The number of circular permutations of n different objects is $(n - 1)!$.

65. If out of n objects,a are alike of one kind, b are alike of another kind, c are alike of third kind and the remaining are all different, then the number of permutations of n objects, taken all at a time, is given be $\frac{n!}{a!b!c!}$.

66. $\binom{n}{r} ={}_n P_r \div r!$

67. ${}_nP_0 = \binom{n}{0} = \binom{n}{n} = 1$

68. ${}_nP_n = n!$

69. $\binom{n}{r} = \binom{n}{n-r}$

70. $\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r}$

71. $(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \binom{n}{2}x^{n-2}a^2 + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$

72. $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$

73. $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1}$

74. $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$

75. $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ if $|x| < 1$

76. $(1-x)^n = 1-nx+\frac{n(n-1)}{2!}x^2-\frac{n(n-1)(n-2)}{3!}x^3+\cdots$ if $|x| < 1$

77. $(1+x)^{-n} = 1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3+\cdots$ if $|x| < 1$

78. $(1-x)^{-n} = 1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots$ if $|x| < 1$

79. $(1-x)^{-\frac{p}{q}} = 1 + \frac{p}{1!}(\frac{x}{q}) + \frac{p(p+q)}{2!}(\frac{x}{q})^2 + \frac{p(p+q)(p+2q)}{3!}(\frac{x}{q})^3 + \cdots$ if $|x| < 1$

80. $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$ if $|x| < 1$

81. $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$ if $|x| < 1$

82. $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$ if $|x| < 1$

83. $(1+x)^{-2} = 1 - 2x + 3x^2 - \cdots$ if $|x| < 1$

84. $(1-x)^{-3} = 1 + \frac{2\cdot 3}{2}x + \frac{3\cdot 4}{2}x^2 + \frac{4\cdot 5}{2}x^3 + \cdots$

85. $(1+x)^{-3} = 1 - \frac{2\cdot 3}{2}x + \frac{3\cdot 4}{2}x^2 - \frac{4\cdot 5}{2}x^3 + \cdots$

86. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

87. $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$

88. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

89. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

90. $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

91. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$ if $|x| \le 1$

92. $\ln(1-x) = -[x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots]$ if $|x| \le 1$

93. $\ln(\frac{1+x}{1-x}) = 2[x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots]$ if $|x| \le 1$

94. $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots$

95. $\ln x = 2[(\frac{x-1}{x+1}) + \frac{1}{3}(\frac{x-1}{x+1})^3 + \cdots]$

96. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

97. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

98. $\tan x = x+\frac{x^3}{3}+\frac{2}{15}x^5-\frac{17}{315}x^7+\frac{62}{2835}x^9+\cdots$

99. $\arcsin x = x + \frac{x^3}{2.3} + \frac{1.3}{2.4.5}x^5 + \frac{1.3.5}{2.4.6.7}x^7 + \cdots$

100. $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

101. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

102. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

103. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

6 Infinite Series

104. $S_n = u_1 + u_2 + \cdots + u_n$ is called the n^{th} partial sum of the series $\sum\limits_{i=1}^{\infty} u_i$.

105. $\sum\limits_{i=1}^{\infty} u_i$ is said to be a convergent series if $\lim\limits_{n\to\infty} S_n$ is finite.

106. $\sum\limits_{i=1}^{\infty} u_i$ is said to be a divergent series if $\lim\limits_{n\to\infty} S_n = \pm\infty$.

107. $\sum\limits_{i=1}^{\infty} u_i$ is said to be oscillatory if S_n does not tend to a definite (finite or infinite) value as n tends to infinity.

108. (Necessary condition for convergence). If a series converges, its n^{th} term approaches zero as n tends to infinity.

109. (p-series) $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

110. (Comparison test) The series $\sum\limits_{i=1}^{\infty} u_i$ converges if $\sum\limits_{i=1}^{\infty} v_i$ is convergent and $0 < u_j \le v_j$ for $j = 1, 2, 3, \cdots$ and diverges if $\sum\limits_{i=1}^{\infty} v_i$ is divergent and $u_j \ge v_j > 0$ for $j = 1, 2, 3, \cdots$

111. (Comparison test-Limit form) $\sum\limits_{i=1}^{\infty} u_i$ and $\sum\limits_{i=1}^{\infty} v_i$ converges (diverges) together if $\lim\limits_{n\to\infty} \frac{u_n}{v_n}$ is finite.

112. (D'Alembert's Ratio Test). $\sum_{i=1}^{\infty} u_i$ of positive terms, converges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} < 1$, diverges if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} > 1$ and this test fails if $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$.

113. (Raabe's Test). $\sum_{i=1}^{\infty} u_i$ of positive terms, converges if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) > 1$, diverges if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) < 1$ and this test fails if $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = 1$.

114. (Logarithmic Test). $\sum_{i=1}^{\infty} u_i$ of positive terms, converges if $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} > 1$, diverges if $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} < 1$ and this test fails if $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = 1$.

115. (Cauchy's Root Test). $\sum_{i=1}^{\infty} u_i$ of positive terms, converges if $\lim_{n\to\infty} u_n^{\frac{1}{n}} < 1$, diverges if $\lim_{n\to\infty} u_n^{\frac{1}{n}} > 1$ and this test fails if $\lim_{n\to\infty} u_n^{\frac{1}{n}} = 1$.

116. An infinite series $\sum_{i=1}^{\infty} (-1)^{i+1} u_i$ whose terms are alternatively positive and negative is called an **alternating series**.

117. (Leibnitz test) An alternating series $\sum_{i=1}^{\infty} (-1)^{i+1} u_i$ converges if (i) $u_j > u_{j+1}$ for all j and (ii) $\lim_{n\to\infty} u_n = 0$.

7 Analytical Geometry(2D)

118. Distance between (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

119. Distance from the origin to (x_1, y_1) is $d = \sqrt{x_1^2 + y_1^2}$

120. The area of a triangle whose vertices are $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ is $\Delta = \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$ $= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

121. The coordinates of the point which divides the line segment joining (x_1, y_1) and (x_2, y_2) internally in the ratio $m : n$ is $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$.

122. The coordinates of the point which divides the line segment joining (x_1, y_1) and (x_2, y_2) externally in the ratio $m : n$ is $\left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-ny_1}{m-n}\right)$.

123. The centroid of the ΔABC, where $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is $G = \left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$.

124. Slope-intercept form of straight line is $y = mx + c$

125. Point-slope form of straight line is $y - y_1 = m(x - x_1)$

126. Two points form of straight line is $\frac{y-y_1}{y_1-y_2} = \frac{x-x_1}{x_1-x_2}$

127. Intercept form of straight line is $\frac{x}{a} + \frac{y}{b} = 1$

128. Normal form of straight line is $x \cos\alpha + y \sin\alpha = p$

129. The acute angle θ between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$ is given by $\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$.

130. The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if $m_1 = m_2$.

131. The lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if $m_1.m_2 = -1$.

132. General equation of straight line is $ax + by + c = 0$; slope $= -\frac{a}{b}$; x-intercept $= -\frac{c}{a}$; y-intercept $= -\frac{c}{b}$

133. The perpendicualr distance from $P(x_1, y_1)$ to the straight line $ax + by + c = 0$ is given by $\left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$.

134. The homogeneous equation of second degree in x and y namely, $ax^2+2hxy+by^2 = 0$ represents a pair of lines intersecting at the origin. If $h^2 > ab$, the lines are real and distinct. If $h^2 = ab$, the lines are real and coincident. If $h^2 < ab$, the lines are imaginary.

135. The acute angle between the pair of lines passing through the origin, $ax^2 + 2hxy + by^2 = 0$ is given by $\cos\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$.

136. The acute angle between the pair of lines $ax^2+2hxy+by^2+2gx+2fy+c=0$ is also $\cos\theta = \pm\frac{2\sqrt{h^2-ab}}{a+b}$.

137. The pair of lines $ax^2+2hxy+by^2=0$ (or $ax^2+2hxy+by^2+2gx+2fy+c=0$) are coincident if $h^2-ab=0$ and they are perpendicular if $a+b=0$.

138. The general second degree equation in x and y namely $ax^2+2hxy+by^2+2gx+2fy+c=0$ will represent a pair of lines if $abc+2fgh-af^2-bg^2-ch^2=0$ or $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$.

139. The point of intersection of lines represented by $ax^2+2hxy+by^2+2gx+2fy+c=0$ is $(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2})$.

140. Equation of the circle with centre at (a,b) and radius r is $(x-a)^2+(y-b)^2=r^2$

141. Equation of the circle with (x_1,y_1) and (x_2,y_2) as extremities of a diameter is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

142. General equation of a circle is $x^2+y^2+2gx+2fy+c=0$ with centre $(-g,-f)$ and radius $\sqrt{g^2+f^2-c}$

143. Equation of the tangent to the circle $x^2+y^2+2gx+2fy+c=0$ on (x_1,y_1) is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

144. Length of the tangent from (x_1,y_1) to the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$

145. The two circles $x^2+y^2+2gx+2fy+c=0$ and $x^2+y^2+2g_1x+2f_1y+c_1=0$ cut orthogonally if $2gg_1+2ff_1=c+c_1$.

146. The locus of a point P which varies in a plane such that its distance from a fixed point S bears a constant ratio to its distance from a fixed line l is called a *conic*. The fixed point S is called the *focus*, the fixed line l the *directrix* and the constant ration e the *eccentriciy* of the conic. The conic is an ellipse,parabola or hyperbola according as $e<1, e=1$ or $e>1$.

147. The equation of second degree in x and y of the form $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a conic. This conic is an ellipse, parabola or hyperbola according as $h^2 < ab, h^2=ab$ or $h^2>ab$.

8 Inequalities

148. For $n \geq 4$, we have $2^n < n! < n^n$.

9 Analytical Geometry(3D)

149. The angle between two lines whose direction ratios are (a_1,b_1,c_1) and (a_2,b_2,c_2) is given by $\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$.

150. If the two lines whose direction ratios are (a_1,b_1,c_1) and (a_2,b_2,c_2) are perpendicular then $a_1a_2+b_1b_2+c_1c_2=0$.

151. Intercept form of a plane : $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

152. General form of a plane : $ax+by+cz+d=0$

153. Equation of the plane passing through (x_1,y_1,z_1) and whose normal's direction ratios are (a,b,c) is given by $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$.

154. Equation of the plance passing through $(x_1,y_1,z_1),(x_2,y_2,z_2)$ and (x_3,y_3,z_3) is given by $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$.

155. Perpendicular distance from (x_1,y_1,z_1) to the plane $ax+by+cz+d=0$ is $\pm\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$

156. Equation of the straight line passing through (x_1,y_1,z_1) with dirction cosines (l,m,n) is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

157. Equation of the straight line passing through (x_1,y_1,z_1) and (x_2,y_2,z_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

158. The condition for two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ may intersect (or) may coplanar is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

159. The projection of the line join of two points (x_1,y_1,z_1) and (x_2,y_2,z_2) on a line whose direction cosines are l,m,n is $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$.

160. Equation of the sphere with centre at (a,b,c) and radius r is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

161. Equation of the sphere with (x_1, y_1, z_1) and (x_2, y_2, z_2) as extremities of a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

162. General equation of a sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$.
Its centre is $(-u, -v, -w)$ and radius is $\sqrt{u^2 + v^2 + w^2 - d}$.

10 Trigonometry

163.

$\theta \rightarrow$	$0°$	$30°$	$45°$	$60°$	$90°$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

164. $\sin(90 \pm \theta) = \cos\theta$

165. $\cos(90 \pm \theta) = \mp \sin\theta$

166. $\tan(90 \pm \theta) = \mp \cot\theta$

167. $\cot(90 \pm \theta) = \mp \tan\theta$

168. $\sec(90 \pm \theta) = \mp \csc\theta$

169. $\csc(90 \pm \theta) = \sec\theta$

170. $\sin(180 \pm \theta) = \mp \sin\theta$

171. $\cos(180 \pm \theta) = -\cos\theta$

172. $\tan(180 \pm \theta) = \pm \tan\theta$

173. $\cot(180 \pm \theta) = \pm \cot\theta$

174. $\sec(180 \pm \theta) = -\sec\theta$

175. $\csc(180 \pm \theta) = \mp \csc\theta$

176. $\sin(270 \pm \theta) = -\cos\theta$

177. $\cos(270 \pm \theta) = \pm \sin\theta$

178. $\tan(270 \pm \theta) = \mp \cot\theta$

179. $\cot(270 \pm \theta) = \mp \tan\theta$

180. $\sec(270 \pm \theta) = \pm \csc\theta$

181. $\csc(270 \pm \theta) = -\sec\theta$

182. $\sin(360 \pm \theta) = \pm \sin\theta$

183. $\cos(360 \pm \theta) = \cos\theta$

184. $\tan(360 \pm \theta) = \pm \tan\theta$

185. $\cot(360 \pm \theta) = \pm \cot\theta$

186. $\sec(360 \pm \theta) = \sec\theta$

187. $\csc(360 \pm \theta) = \pm \csc\theta$

188. $e^{i\theta} = \cos\theta + i\sin\theta$

189. $\cos\theta = \frac{(e^{i\theta} + e^{-i\theta})}{2}$

190. $\sin\theta = \frac{(e^{i\theta} - e^{-i\theta})}{2i}$

191. $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

192. $\sin^2\theta + \cos^2\theta = 1$

193. $1 + \tan^2\theta = \sec^2\theta$

194. $1 + \cot^2\theta = \csc^2\theta$

195. $\sin 2\theta = 2\sin\theta\cos\theta$

196. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

197. $\cos 2\theta = 1 - 2\sin^2\theta$

198. $\cos 2\theta = 2\cos^2\theta - 1$

199. $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

200. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

201. $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

202. $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

203. $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

204. $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

205. $\cos(A + B)\cos(A - B) = \cos^2 A - \sin^2 B$

206. $2\sin A\cos B = \sin(A + B) + \sin(A - B)$

207. $2\cos A\sin B = \sin(A + B) - \sin(A - B)$

208. $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

209. $-2\sin A\sin B = \cos(A + B) - \cos(A - B)$

210. $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

211. $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

212. $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

213. $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

214. $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$

215. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$

216. $\sin 2A = \frac{2\tan A}{1 + \tan^2 A}$

217. $\cos 2A = \frac{1 - \tan^2 A}{1 + \tan^2 A}$

218. $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$

219. $\cos n\theta = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta$
$+\binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots$

220. $\sin n\theta = \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + \cdots$

221. $\sin^{-1}(\sin\theta) = \theta = \sin(\sin^{-1}\theta)$

222. $\cos^{-1}(\cos\theta) = \theta = \cos(\cos^{-1}\theta)$

223. $\tan^{-1}(\tan\theta) = \theta = \tan(\tan^{-1}\theta)$

224. $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

225. $\sec^{-1}x + \csc^{-1}x = \frac{\pi}{2}$

226. $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$

227. $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$
$= \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \csc^{-1}\frac{1}{x}$
$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$

228. $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2}$
$= \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x}$
$= \csc^{-1}\frac{1}{\sqrt{1-x^2}} = \tan^{-1}\frac{\sqrt{1-x^2}}{x}$

229. $\sin^{-1}x + \sin^{-1}y =$
$\sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

230. $\sin^{-1}x - \sin^{-1}y =$
$\sin^{-1}\left[x\sqrt{1-y^2} - y\sqrt{1-x^2}\right]$

231. $\cos^{-1}x + \cos^{-1}y =$
$\cos^{-1}\left[xy - \sqrt{1-x^2}\sqrt{1-y^2}\right]$

232. $\cos^{-1}x - \cos^{-1}y =$
$\cos^{-1}\left[xy + \sqrt{1-x^2}\sqrt{1-y^2}\right]$

233. $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

234. $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left[\frac{x-y}{1+xy}\right]$

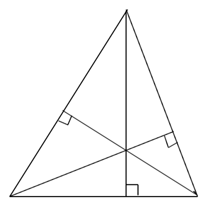
235. $2\tan^{-1}x = \tan^{-1}\left[\frac{2x}{1-x^2}\right]$

236. $\log(a+ib) = \frac{1}{2}\log(a^2+b^2) + i\tan^{-1}(\frac{b}{a})$

11 Solution of Triangle

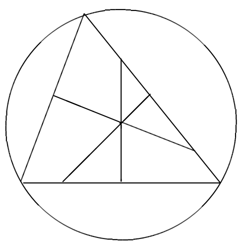
Notation: A, B, C are angles of ΔABC
a, b, c are oppposite sides to the angles A, B, C respectively.
r- radius of in-circle;
R- radius of Circum-circle;
r_1, r_2, r_3 are radii of ex-circles opposite to A, B, C;
s - semi-perimeter.
Δ - area of ΔABC. m_a, m_b, m_c - length of Medians through A, B, C respectively.

237. The orthocenter of a triangle is the intersection of the triangle's altitudes.



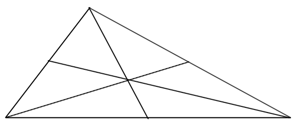
Orthocenter

238. The circumcenter of a triangle is the center of the circumscribed circle (the intersection of the perpendicular bisectors of the three sides).



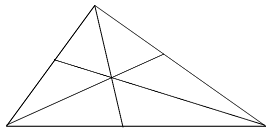
Circumcenter

239. The centroid of a triangle is the intersection of the three medians of the triangle.



Centroid

240. The incenter of a triangle is the intersection of the angle bisectors of the triangle.


Incenter

241. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

242. $\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$

243. $\Delta = 2R^2 \sin A \sin B \sin C$

244. $\Delta = \frac{abc}{4R}$

245. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

246. $a^2 = b^2 + c^2 - 2bc\cos A$

247. $b^2 = c^2 + a^2 - 2ca\cos B$

248. $c^2 = a^2 + b^2 - 2ab\cos C$

249. $a = b\cos C + c\cos B$

250. $b = c\cos A + a\cos C$

251. $c = a\cos B + b\cos A$

252. $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$

253. $\tan\frac{C-A}{2} = \frac{c-a}{c+a}\cot\frac{B}{2}$

254. $\tan\frac{A-B}{2} = \frac{a-b}{a+b}\cot\frac{C}{2}$

255. $\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$

256. $\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$

257. $\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$

258. $\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$

259. $\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$

260. $\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$

261. $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

262. $\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}$

263. $\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$

264. $r = \frac{\Delta}{s}$

265. $r = (s-a)\tan\frac{A}{2}$

266. $r = (s-b)\tan\frac{B}{2}$

267. $r = (s-c)\tan\frac{C}{2}$

268. $r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2}$

269. $r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2}$

270. $r_2 = \frac{\Delta}{s-c} = s\tan\frac{C}{2}$

271. $r_1 = (s-c)\cot\frac{B}{2} = (s-b)\cot\frac{C}{2}$

272. $r_2 = (s-c)\cot\frac{A}{2} = (s-a)\cot\frac{C}{2}$

273. $r_3 = (s-a)\cot\frac{B}{2} = (s-b)\cot\frac{A}{2}$

274. Median $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

275. Median $m_b = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2}$

276. Median $m_c = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$

277. $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

12 Hyperbolic functions

278. $\sinh x = \frac{e^x - e^{-x}}{2}$; $\cosh x = \frac{e^x + e^{-x}}{2}$

279. $\tanh x = \frac{\sinh x}{coshx}$; $\coth x = \frac{\cosh x}{\sinh x}$

280. $\text{csc}hx = \frac{1}{\sinh x}$; $\text{sec}hx = \frac{1}{\cosh x}$

281. $\sin(ix) = i\sinh x$

282. $\cos(ix) = \cosh x$

283. $\tan(ix) = i\tanh x$

284. $\cosh^2 x - \sinh^2 x = 1$

285. $1 - \tanh^2 x = \text{sec}h^2 x$

286. $\coth^2 x - 1 = \text{csc}h^2 x$

287. $\sinh(x \pm y) = \sinh x\cosh y \pm \cosh x\sinh y$

288. $\cosh(x \pm y) = \cosh x\cosh y \pm \sinh x\sinh y$

289. $\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x\tanh y}$

290. $\sinh 2x = 2\sinh x\cosh x$

291. $\cosh 2x = \cosh^2 x + \sinh^2 x$

292. $\cosh 2x = 2\cosh^2 x - 1$

293. $\cosh 2x = 1 + 2\sinh^2 x$

294. $\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

295. $\sinh 3x = 3\sinh x + 4\sinh^3 x$

296. $\cosh 3x = 4\cosh^3 x - 3\cosh x$

297. $\tanh 3x = \frac{3\tanh x + \tanh^3 x}{1+3\tanh^2 x}$

298. $\sinh x + \sinh y = 2\sinh\frac{x+y}{2}\cosh\frac{x-y}{2}$

299. $\sinh x - \sinh y = 2\cosh\frac{x+y}{2}\sinh\frac{x-y}{2}$

300. $\cosh x + \cosh y = 2\cosh\frac{x+y}{2}\cosh\frac{x-y}{2}$

301. $\cosh x - \cosh y = 2\sinh\frac{x+y}{2}\sinh\frac{x-y}{2}$

302. $\sinh^{-1} x = \ln(x+\sqrt{x^2+1})$

303. $\cosh^{-1} x = \ln(x+\sqrt{x^2-1})$

304. $\tanh^{-1} x = \frac{1}{2}\ln(\frac{1+x}{1-x})$

13 Vector Calculus

305. If α, β, γ are the angles made by $\vec{r}$ with the co-ordinate axes OX, OY and OZ then $\cos\alpha, \cos\beta, \cos\gamma$ are called the direction cosines of $\vec{r}$. Further $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

306. The scalar product (or Dot product) of two vectors $\vec{a}$ and $\vec{b}$ is defined as the number $|\vec{a}||\vec{b}|\cos\theta$, where θ is the angle between $\vec{a}$ and $\vec{b}$. It is denoted by $\vec{a}\,.\,\vec{b}$

307. grad $\phi = \nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$

308. Div $\vec{F} = \nabla.\,\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

309. Div $\vec{F} = \vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}$

310. Curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

311. Curl $\vec{F} = \vec{i}\times\frac{\partial\vec{F}}{\partial x} + \vec{j}\times\frac{\partial\vec{F}}{\partial y} + \vec{k}\times\frac{\partial\vec{F}}{\partial z}$

312. $\vec{F}$ is solenoidal if $\nabla.\,\vec{F} = 0$

313. $\vec{F}$ is irrotational if $\nabla\times\vec{F} = 0$

314. grad $(f \pm g)$ = grad (f) $\pm$ grad (g) .

315. grad (fg) = f grad (g) + g grad (f) .

316. Directional derivative = grad(f) $\cdot\,\hat{n}$

14 Limits

317. Indeterminate forms of limits:
$\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0.\infty, 0^0, 1^\infty, \infty^0$

318. The following are determinate forms:
$\infty + \infty \to \infty$
$-\infty - \infty \to -\infty$
$(0^+)^\infty \to 0$
$(0^+)^{-\infty} \to \infty$
$\frac{0}{\infty} \to 0$, $\frac{\infty}{0^+} \to \infty$ and $\frac{\infty}{0^-} \to -\infty$

319. $\lim\limits_{x\to a}\frac{x^n - a^n}{x-a} = na^{n-1}$

320. $\lim\limits_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

321. $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$

322. $\lim\limits_{n\to\infty} n^{\frac{1}{n}} = 1$

15 Differentiation

323. $\frac{d}{dx}(x^n) = nx^{n-1}$

324. $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

325. $\frac{d}{dx}(a^x) = a^x\ \ln a$

326. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

327. $\frac{d}{dx}(e^x) = e^x$

328. $\frac{d}{dx}(\sin x) = \cos x$

329. $\frac{d}{dx}(\cos x) = -\sin x$

330. $\frac{d}{dx}(\tan x) = \sec^2 x$

331. $\frac{d}{dx}(\csc x) = -\csc x\cot x$

332. $\frac{d}{dx}(\sec x) = \sec x\tan x$

333. $\frac{d}{dx}(\cot x) = -\csc^2 x$

334. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

335. $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$

336. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

337. $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

338. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

339. $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

340. $\frac{d}{dx}(\sinh x) = \cosh x$

341. $\frac{d}{dx}(\cosh x) = \sinh x$

342. $\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

343. $\frac{d}{dx}(\text{csch} x) = -\text{csch} x \coth x$

344. $\frac{d}{dx}(\text{sech} x) = -\text{sech} x \tanh x$

345. $\frac{d}{dx}(\text{coth} x) = -\text{csch}^2 x$

346. $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$

347. $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

348. $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$

349. $\frac{d}{dx}(uv) = \frac{du}{dx}\, v + u\, \frac{dv}{dx}$

350. $\frac{d}{dx}(uvw) = \frac{du}{dx}\, v\, w + u\, \frac{dv}{dx}\, w + u\, v\, \frac{dw}{dx}$

351. $\frac{d}{dx}(\frac{u}{v}) = \frac{v\,\frac{du}{dx} - u\,\frac{dv}{dx}}{v^2}$

352. $\frac{dy}{dx} = \frac{dy}{du}\, \frac{du}{dx}$

353. $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

354. $D^n(ax+b)^m = \frac{m!a^n(ax+b)^{m-n}}{(m-n)!}$

355. $D^n(ax+b)^{-1} = (-1)^n n! a^n (ax+b)^{-n-1}$

356. $D^n \ln(ax+b) = (-1)^{n-1}(n-1)!a^n(ax+b)^{-n}$

357. $D^n \sin(ax+b) = a^n \sin(\frac{n\pi}{2} + ax + b)$

358. $D^n \cos(ax+b) = a^n \cos(\frac{n\pi}{2} + ax + b)$

359. $D^n(e^{ax}\sin(bx+c)) = r^n e^{ax} \sin(bx+c+n\phi)$ where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$

360. $D^n(e^{ax}\cos(bx+c)) = r^n e^{ax} \cos(bx+c+n\phi)$ where $r = \sqrt{a^2+b^2}$ and $\phi = \tan^{-1}(\frac{b}{a})$

361. $D^n(u\,v) = D^n u \cdot v + \binom{n}{1} D^{n-1} u \cdot Dv + \binom{n}{2} D^{n-2} u \cdot D^2 v + \cdots + u \cdot D^n v$

16 Differential Calculus

362. The radius of curvature of $y = f(x)$ is

$$\rho = \frac{\left[1 + (\frac{dy}{dx})^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + (\frac{dx}{dy})^2\right]^{\frac{3}{2}}}{\frac{d^2x}{dy^2}}$$

363. Curvature $= \frac{1}{\rho}$

364. The radius of curvature of $x = x(t), y = y(t)$ at any point t is given by

$$\rho = \frac{\left[(x')^2 + (y')^2\right]^{\frac{3}{2}}}{x'y'' - y'x''}$$

365. Circle of curvature of $y = f(x)$ is $(x - \overline{x})^2 + (y - \overline{y})^2 = \rho^2$ where

$$\overline{x} = x - \frac{y_1(1+y_1^2)^{\frac{3}{2}}}{y_2}, \overline{y} = y + \frac{1+y_1^2}{y_2},$$

$$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}$$

366. Rolle's Theorem: If $f(x)$ is continuous in the closed interval $a \le x \le b$ and if $f'(x)$ exists in the open interval $a < x < b$ and if $f(x)$ is zero when $x = a$ and $x = b$, then $f'(x)$ will be zero for at least one value of x between a and b.

367. Mean Value Theorem: If $f(x)$ is continuous in the closed interval $a \le x \le b$ and $f'(x)$ exists in the open interval $a < x < b$, then there is at least one value of x, say x_1, between a and b such that $f'(x_1) = \frac{f(b)-f(a)}{b-a}$.

17 Integration

368. $\int x^n dx = \frac{x^{n+1}}{n+1} + c \ (n \neq -1)$

369. $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$

370. $\int \frac{1}{x} dx = lnx + c$

371. $\int e^x dx = e^x + c$

372. $\int a^x dx = \frac{a^x}{\ln a} + c$

373. $\int \sin x dx = -\cos x + c$

374. $\int \cos x dx = \sin x + c$

375. $\int \tan x dx = \ln \sec x + c$

376. $\int \csc x dx = \ln(\csc x - \cot x) + c$

377. $\int \sec x dx = \ln(\sec x + \tan x) + c$

378. $\int \cot x dx = \ln \sin x + c$

379. $\int \sec^2 x dx = \tan x + c$

380. $\int \csc^2 x dx = -\cot x + c$

381. $\int \sec x \tan x dx = \sec x + c$

382. $\int \csc x \cot x dx = -\csc x + c$

383. $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} \ln(\frac{a+x}{a-x}) + c$

384. $\int \frac{1}{a^2-x^2} dx = \frac{1}{a} \tanh^{-1}(\frac{x}{a}) + c$

385. $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln(\frac{x-a}{x+a}) + c$

386. $\int \frac{1}{x^2-a^2} dx = -\frac{1}{a} \coth^{-1}(\frac{x}{a}) + c$

387. $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$

388. $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a}) + c$

389. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \sinh^{-1}(\frac{x}{a}) + c$

390. $\int \frac{1}{\sqrt{a^2+x^2}} dx = \ln(x + \sqrt{a^2 + x^2}) + c$

391. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \cosh^{-1}(\frac{x}{a}) + c$

392. $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln(x + \sqrt{x^2 - a^2}) + c$

393. $\int \sqrt{a^2 - x^2} dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}(\frac{x}{a}) + c$

394. $\int \sqrt{a^2 + x^2} dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2} \sinh^{-1}(\frac{x}{a}) + c$

395. $\int \sqrt{x^2 - a^2} dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2} \cosh^{-1}(\frac{x}{a}) + c$

396. $\int \sinh x dx = \cosh x + c$

397. $\int \cosh x dx = \sinh x + c$

398. $\int \text{sech}^2 x dx = \tanh x + c$

399. $\int \text{csch}^2 x dx = -\coth x + c$

400. $\int \text{sech} x \tanh x dx = -\text{sech} x + c$

401. $\int \text{csch} x \coth x dx = -\text{csch} x + c$

402. $\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

403. $\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

404. $\int u\, dv = uv - \int v\, du + c$

405. $\int u\, dv = uv - u'v_1 + u''v_2 - \cdots$

406. $\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + c \ (n \neq -1)$

407. $\int \frac{f'(x)}{f(x)} dx = \ln f(x) + c$

408. $\int F(f(x)) f'(x) dx = \int F(u) du$
where u = f(x)

409. $\int_a^b f(x) dx = -\int_b^a f(x) dx$

410. $\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx, a \leq b \leq c$

411. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

412. $\int_{-a}^a f(x) dx = 0$ if $f(x)$ is an odd function

413. $\int_{-a}^a f(x) dx = 2\int_0^a f(x) dx$ if $f(x)$ is an even function

414. $\int_0^{2a} f(x) dx = 2\int_0^a f(x) dx$ if $f(2a-x) = f(x)$

415. $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

416. If $I_n = \int (1-x^3)^n dx$, where $n = 0, 1, 2, \cdots$ then $(3n+1) I_n = x(1-x^3)^n + 3n I_{n-1}$

417. If $I_n = \int \sin^n x dx$, where $n = 1, 2, \cdots$ then $nI_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2}$.

418. If $I_n = \int \cos^n x dx$, where $n = 1, 2, \cdots$ then $nI_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$.

419. If $I_{m,n} = \int \sin^m x \cos^n x dx, m, n = 1, 2, \cdots$ then

$$(m+n) I_{m,n} = \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2}$$

420. $\int_0^{\frac{\pi}{2}} \sin^n x dx = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \lambda$ where $\lambda = 1$ if n is odd and $\lambda = \frac{\pi}{2}$ if n is even

421. $\int_0^{\frac{\pi}{2}} \cos^n x dx = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \cdots \lambda$ where $\lambda = 1$ if n is odd and $\lambda = \frac{\pi}{2}$ if n is even

422. $\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)\cdots \times (n-1)(n-3)\cdots}{(m+n)(m+n-2)\cdots} \lambda$ where $\lambda = \frac{\pi}{2}$ if both m and n are even, Otherwise $\lambda = 1$

423. $\frac{d}{d\alpha} \int_{\phi(\alpha)}^{\psi(\alpha)} f(x, \alpha) dx = \frac{d\psi}{d\alpha} f(\psi, \alpha) - \frac{d\phi}{d\alpha} f(\phi, \alpha) + \int_{\phi(\alpha)}^{\psi(\alpha)} \frac{\partial f}{\partial \alpha} dx$

424. Area bounded by the curve $y = f(x)$, x-axis, and the ordinates $x = a$ and $x = b$ is $\int_a^b y dx$.

425. Length of the arc of $y = f(x)$ from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$.

426. Volume generated by the revolution of area between the curve $y = f(x)$, x-axis, and the ordinates $x = a$ and $x = b$ is $\int_a^b \pi y^2 dx$.

427. The surface area of the solid, obtained by revolving about the x-axis the arc of $y = f(x)$ intercepted between the ordinates $x = a$ and $x = b$ is $\int_a^b 2\pi y\sqrt{1 + (\frac{dy}{dx})^2}dx$.

18 Improper Integrals

428. $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$

429. $\Gamma(n+1) = n\Gamma(n)$

430. $\Gamma(n+1) = n!$ if n is a $+ve$ integer

431. $\Gamma(\frac{1}{2}) = \sqrt{\pi}; \Gamma(0) = \infty$

432. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, m, n > 0$

433. $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

434. $\beta(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}dy$

435. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

436. $\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\beta(\frac{p+1}{2}, \frac{q+1}{2})$

437. $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^n\theta \, d\theta = \frac{1}{2}\beta(\frac{n+1}{2}, \frac{1}{2})$

19 Laplace Transform

438. $L[f(t)] = \int_0^\infty e^{-st}f(t)\,dt = F(s)$

439. $L[af(t) + bg(t)] = aL[f(t)] + bL[g(t)]$

440. $L[t^n] = \frac{\Gamma(n+1)}{s^{n+1}}$

441. $L[1] = \frac{1}{s}$; $L[t] = \frac{1}{s^2}$; $L[t^2] = \frac{2}{s^3}$

442. $L[e^{-at}] = \frac{1}{s+a}$; $L[e^{at}] = \frac{1}{s-a}$

443. $L[\sinh at] = \frac{a}{s^2-a^2}$; $L[\cosh at] = \frac{s}{s^2-a^2}$

444. $L[\sin at] = \frac{a}{s^2+a^2}$; $L[\cos at] = \frac{s}{s^2+a^2}$

445. $L[f(at)] = \frac{1}{a}F(\frac{s}{a})$

446. $L[e^{-at}f(t)] = F(s+a)$

447. $L[e^{at}f(t)] = F(s-a)$

448. $L[t\,f(t)] = -F'(s)$

449. $L[t^n\,f(t)] = (-1)^nF^{(n)}(s)$

450. $L[t\sin at] = \frac{2as}{(s^2+a^2)^2}$

451. $L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$

452. $L[\frac{f(t)}{t}] = \int_s^\infty F(s)ds$, if $\lim_{t\to 0} \frac{f(t)}{t}$, exists.

453. $L[f'(t)] = sF(s) - sf(0)$

454. $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

455. $L[\int_0^t f(t)dt] = \frac{L[f(t)]}{s}$

456. $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

457. $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

458. $L[f(t)] = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$
if $f(p+t) = f(t)$.

459. Convolution: $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

460. $L[f(t) * g(t)] = L[f(t)]\,L[g(t)]$

461. If $L[f(t)] = F(s)$ then $L^{-1}[F(s)] = f(t)$.

462. $L^{-1}[\frac{1}{s}] = 1$; $L^{-1}[\frac{1}{s^2}] = t$

463. $L^{-1}[\frac{1}{s^{n+1}}] = \frac{t^n}{n!}$

464. $L^{-1}[\frac{1}{s-a}] = e^{at}$; $L^{-1}[\frac{1}{s+a}] = e^{-at}$

465. $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \sinh at$; $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

466. $L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$; $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

467. $L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]$

468. $L^{-1}[F(s-a)] = e^{at}L^{-1}[F(s)]$

469. $L^{-1}[\frac{2as}{(s^2+a^2)^2}] = t\sin at$

470. $L^{-1}[\frac{s^2-a^2}{(s^2+a^2)^2}] = t\cos at$

471. $L^{-1}[F'(s)] = -tL^{-1}[F(s)]$

472. $L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt$

473. $L^{-1}[F(s)\,G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$

20 Z - Transform

474. $Z[\{f(n)\}] = \sum_{n=0}^{\infty} f(n)z^{-n} = F(z)$

475. $Z[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} = F(z)$

476. $Z[1] = \frac{z}{z-1}$; $Z[(-1)^n] = \frac{z}{z+1}$

477. $Z[a^n] = \frac{z}{z-a}$

478. $Z[e^{an}] = \frac{z}{z-e^a}$; $Z[e^{-an}] = \frac{z}{z-e^{-a}}$

479. $Z\left[\frac{a^n}{n!}\right] = e^{\frac{a}{z}}$; $Z\left[\frac{1}{n!}\right] = e^{\frac{1}{z}}$

480. $Z[n] = \frac{z}{(z-1)^2}$; $Z[na^n] = \frac{az}{(z-a)^2}$

481. $Z[a^n f(n)] = F(\frac{z}{a})$ where $F(z) = Z[f(n)]$

482. $Z[e^{-at}f(t)] = F(ze^{aT})$ where $F(z) = Z[f(t)]$

483. $Z[e^{at}f(t)] = F(ze^{-aT})$ where $F(z) = Z[f(t)]$

484. $Z[\cos\frac{n\pi}{2}] = \frac{z^2}{z^2+1}$; $Z[\sin\frac{n\pi}{2}] = \frac{z}{z^2+1}$

485. $Z[\cos at] = \frac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

486. $Z[\sin at] = \frac{z\sin aT}{z^2-2z\cos aT+1}$

487. If $Z[f(n)] = F(s)$ then $Z[nf(n)] = -z\frac{d}{dz}F(z)$

488. $Z[f(n-k)] = z^{-k}Z[f(n)]$

489. $Z[f(n+k)] = z^k\left[Z[f(n)] - \sum_{n=0}^{k-1} f(n)z^{-n}\right]$

490. $f(n) = Z^{-1}[F(z)] = \int_C z^{n-1}F(z)dz$

491. $Z^{-1}[F(z)G(z)] = Z^{-1}[F(z)] * Z^{-1}[G(z)]$

21 Fourier Series

492. The Fourier series of $f(x)$ in $(c, c+2l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l})$ where $a_0 = \frac{1}{l}\int_c^{c+2l} f(x)dx$;
$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\frac{n\pi x}{l}dx$
$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}dx$

493. The Fourier series of $f(x)$ in $(0, 2\pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx)$ where
$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$;
$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$
$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$

494. The Fourier series of $f(x)$ in $(-\pi, \pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n\cos nx + b_n\sin nx)$ where
$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$;
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$
$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

495. The half range Fourier Cosine series of $f(x)$ in $(0, l)$ is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n\cos\frac{n\pi x}{l}$ where
$a_0 = \frac{2}{l}\int_0^l f(x)dx$;
$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$

496. The half range Fourier sine series of $f(x)$ in $(0, l)$ is $f(x) = \sum_{n=0}^{\infty} b_n\sin\frac{n\pi x}{l}$ where
$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$

497. The half range Fourier Cosine series of $f(x)$ in $(0, \pi)$ is $f(x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n\cos nx$ where
$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$;
$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx$

498. The half range Fourier sine series of $f(x)$ in $(0, \pi)$ is $f(x) = \sum_{n=0}^{\infty} b_n\sin nx$ where
$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$

499. The complex form of Fourier series of $f(x)$ in $(c, c+2l)$ is $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$ where
$c_n = \frac{1}{2l}\int_c^{c+2l} f(x)e^{-\frac{in\pi x}{l}}dx.$

22 Numerical Methods

500. **Intermediate Theorem:** Let $f(x)$ be a continuous function on $[a,b]$. If $d \in [f(a), f(b)]$ then there exists a $c \in [a,b]$ such that $f(c) = d$.

501. $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where a_i's are constants, is called an *Algebraic equation* or a *Polynomial equation* of n^{th} degree, if $a_0 \neq 0$. A non-algerbraic equation is called a *Transcendental equation.*

502. If $f(x) = 0$ is continuous in $a \leq x \leq b$ and if $f(a)$ and $f(b)$ are of opposite signs, the $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

503. Every algebraic equation of odd degree has at least one real root whose sign is opposite to that of its last term.

504. Every algebraic equation of even degree with last term negative, has at least a pair of real roots one is positive and the other negative.

505. **Iterative Method :** To find a real root of $f(x) = 0$ by iterative method, rewrite the given equation in the form of $x = \phi(x)$ where $|\phi'(x)| < 1$ in (a,b). (Note: $f(a)$ and $f(b)$ are of opposite signs).

506. **Regula-Falsi Method:** To find a real root of $f(x) = 0$, find two real numbers a and b such that $f(a)$ and $f(b)$ are of opposite signs. The first approximate root is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$.

507. **Newton-Raphson Method**(single variable): To find a real root of $f(x) = 0$ by Newton-Raphson method, the first approximate root is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ where x_0 is an initial approximate root of $f(x) = 0$. In general $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. (Note: Choose $x_0 = \frac{a+b}{2}$ if $f(a)$ and $f(b)$ are of opposite signs).

508. The order of convergence of Newton-Raphson method is two.

509. The condition for convergence of Newton-Raphson method is $|f(x)f''(x)| < |f'(x)|^2$.

510. **Newton-Raphson Method**(Two variables): Consider the system of equations $f(x,y) = 0$ and $g(x,y) = 0$ with the initial approximation being (x_0, y_0). The first approximate root is given by $x_1 = x_0 + h, y_1 = y_0 + k$ where $h = -\frac{D_1}{D}, k = -\frac{D_2}{D}$, $D = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}_{(x_0,y_0)}, D_1 = \begin{vmatrix} f & f_y \\ g & g_y \end{vmatrix}_{(x_0,y_0)}$, $D_2 = \begin{vmatrix} f_x & f \\ g_x & g \end{vmatrix}_{(x_0,y_0)}$

511. **Gauss Elimination Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss Elimination method,* transform the augmented matrix $[A,B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ into an upper triangular matrix by the elementary row operations. Back substitution on the transformed equations will give the required solution.

512. **Gauss-Jordan Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss-Jordan method,* transform the augmented matrix $[A,B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ into a diagonal matrix by the elementary row operations. The transformed equations will give the required solution.

513. A square matrix is said to be *diagonally dominant* if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical values of other elements in that row.

514. **Gauss-Jacobi Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x+b_2y+c_2z = d_2, a_3x+b_3y+c_3z = d_3$ by *Gauss-Jordan method.* Assume that the coefficient matrix is diagonally dominant. Rewrite the given system of equations as $x = \frac{1}{a_1}(d_1 - b_1y - c_1z); y = \frac{1}{b_2}(d_2 - a_2x - c_2z); z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$. If $x^{(0)}, y^{(0)}, z^{(0)}$ be the initial guess for x, y, z respectively, then the first approximate root is given by $x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}); y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}); z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)})$. Using these values, we get the second approximate root as $x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)}); y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(1)}); z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$. Repeating this process, we get the required solution.

515. **Gauss-Seidel Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x+b_2y+c_2z = d_2, a_3x+b_3y+c_3z = d_3$ by *Gauss-Seidel method*. Assume that the coefficient matrix is diagonally dominant. Rewrite the given system of equations as $x = \frac{1}{a_1}(d_1 - b_1y - c_1z); y = \frac{1}{b_2}(d_2 - a_2x - c_2z); z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$. If $y^{(0)}, z^{(0)}$ be the initial guess for y, z respectively, then the first approximate root is given by $x^{(1)} = \frac{1}{a_1}(d_1-b_1y^{(0)}-c_1z^{(0)}); y^{(1)} = \frac{1}{b_2}(d_2-a_2x^{(1)}-c_2z^{(0)}); z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$. Using these values, we get the second approximate root as $x^{(2)} = \frac{1}{a_1}(d_1-b_1y^{(1)}-c_1z^{(1)}); y^{(2)} = \frac{1}{b_2}(d_2-a_2x^{(2)}-c_2z^{(1)}); z^{(2)} = \frac{1}{c_3}(d_3-a_3x^{(2)}-b_3y^{(2)})$. Repeating this process, we get the required solution.

516. **Power Method:** To find the numerically largest eigen value and the corresponding eigen vector of a square matrix A by Power method, choose the initial eigen vector as $X_0 = (1, 0, 0)'$. Then
$AX_0 = \lambda_1 X_1$ where $X_1 = (1, a_1, b_1)'$
$AX_1 = \lambda_2 X_2$ where $X_2 = (1, a_2, b_2)'$
$AX_2 = \lambda_3 X_3$ where $X_3 = (1, a_3, b_3)'$
The sequence $\lambda_1, \lambda_2, \lambda_3, \cdots$ converges to the numerically largest eigen value of A and the sequence $X_1, X_2, X_3, \cdots$ converges to the corresponding eigen vector.

517. **Finite Differences:** Let $y = f(x)$ be a given function of x and let $y_i = f(x_i)$ for $i = 0, 1, 2, \cdots, n$ where $x_i = x_{i-1} + h$ for $i = 1, 2, \cdots, n$, h is the interval of differencing. Now $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ are called the first differences of the function y. We denote $\Delta y_i = y_{i+1} - y_i$, Δ is called the forward difference operator. Now the differences of these first differences are called second differences. Thus $\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i) = \Delta y_{i+1} - \Delta y_i = (y_{i+2}-y_{i+1})-(y_{i+1}-y_i) = y_{i+2}-2y_{i+1}+y_i$. In general, $\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$.

518. $\Delta f(x) = f(a + h) - f(x)$.

519. **Forward difference table**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

520. **Newton's Forward Interpolation Formula**
$y_n = y(x_0+nh) = y_0+n\Delta y_0+\frac{n(n-1)}{2!}\Delta^2 y_0+\frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$
(This formula gives greater accuracy when $x_0 + nh$ is near the beginning of the table)

521. **Backward Difference:** The backward difference operator ∇ is defined as $\nabla y_i = y_i - y_{i-1}$. The second backward difference is $\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1} = (y_i - y_{i-1}) - (y_{i-1} - y_{i-2}) = y_i - 2y_{i-1} + y_{i-2}$. In general $\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$.

522. $\nabla f(x) = f(x) - f(x - h)$.

523. **Backward difference table**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_{-4}	y_{-4}				
		∇y_{-3}			
x_{-3}	y_{-3}		$\nabla^2 y_{-2}$		
		∇y_{-2}		$\nabla^3 y_{-1}$	
x_{-2}	y_{-2}		$\nabla^2 y_{-1}$		$\nabla^4 y_0$
		∇y_{-1}		$\nabla^3 y_0$	
x_{-1}	y_{-1}		$\nabla^2 y_0$		
		∇y_0			
x_0	y_0				

524. **Newton's Backward Interpolation Formula**
$y_n = y(x_0+nh) = y_0+n\nabla y_0+\frac{n(n+1)}{2!}\nabla^2 y_0+\frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \cdots$
(This formula gives greater accuracy when $x_0 + nh$ is near the end of the table)

525. **Central difference:** The central difference operator δ is defined as $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$.

526. $\delta y_x = \Delta y_{x-\frac{h}{2}}$ or $\Delta y_x = \delta y_{x+\frac{h}{2}}$.

527. **Central difference table**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		$\mathbf{\Delta y_{-1}}$		$\mathbf{\Delta^3 y_{-2}}$	
$\mathbf{x_0}$	$\mathbf{y_0}$		$\mathbf{\Delta^2 y_{-1}}$		$\mathbf{\Delta^4 y_{-2}}$
		$\mathbf{\Delta y_0}$		$\mathbf{\Delta^3 y_{-1}}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

528. **Stirling's Central difference formula**
$y(x_0+nh) = y_0+n\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right)+\frac{n^2}{2!}\Delta^2 y_{-1}+\frac{n(n^2-1)}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right)+\frac{n^2(n^2-1)}{4!}\Delta^4 y_{-2}+\cdots$

529. **Divided Differences:** Let $y = f(x)$ be the given function. Let $f(x_0), f(x_1), \cdots, f(x_n)$ be the values of the function corresponding to the arguments $x_0, x_1, \cdots, x_n$ and $x_0, x_1, \cdots, x_n$ need not be equally spaced. The first divided difference of $f(x)$ for the arguments x_0 and x_1 is defined as $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$. Similarly, $f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. The second divided difference of $f(x)$ for the arguments x_0, x_1 and x_2 is $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$.

530. **Lagranges Interpolation formula:** Let $y = f(x)$ be the given function.
Let $f(x_0), f(x_1), \cdots, f(x_n)$ be the values of the function corresponding to the arguments $x_0, x_1, \cdots, x_n$ and $x_0, x_1, \cdots, x_n$ need not be equally spaced.
$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}f(x_1) + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$ is called the Lagrange's interpolation formula.

531. **Numerical Differentiation:** Find the polynomial from the given data using Newton's Forward/divided difference interpolation formula and hence find its derivative at the desired point.

532. Newton's forward interpolation formula to find the derivative at $x = x_0$ is
$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$
and $f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$

533. Newton's backward interpolation formula to find the derivative at $x = x_0$ is
$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{\nabla^2 y_0}{2} + \frac{\nabla^3 y_0}{3} + \frac{\nabla^4 y_0}{4} + \cdots\right]$
and $f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \cdots\right]$

534. **Trapezoidal rule:**
$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$
where $h = \frac{b-a}{n}$, n- Number of intervals; $x_0 = a;\ x_n = b;\ x_i = x_{i-1} + h;\ y_i = f(x_i)$

535. **Simpson's $\frac{1}{3}^{rd}$ rule:**
$\int_a^b f(x)dx = \frac{h}{3}[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)]$ where $h = \frac{b-a}{n}$, n- Number of intervals(even); $x_0 = a;\ x_n = b;\ x_i = x_{i-1} + h;\ y_i = f(x_i)$

536. **Simpson's $\frac{3}{8}^{th}$ rule:**
$\int_a^b f(x)dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5+\cdots)+2(y_3+y_6+\cdots)]$ where $h = \frac{b-a}{n}$, n- Number of intervals(a multiple of 3); $x_0 = a;\ x_n = b;\ x_i = x_{i-1} + h;\ y_i = f(x_i)$

537. **Gaussian two point quadrature formula:**
$\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}}) + f(\frac{1}{\sqrt{3}})$

538. **Gaussian three point quadrature formula:**
$\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$

539. To evaluate $\int_a^b f(x)dx$ by Gaussian quadrature formula, put $t = \frac{(b-a)x+(b+a)}{2}$ then apply Gaussian quadrature formula.

540. **Double integral using Trapezoidal rule:**
$\int_c^d\int_a^b f(x, y)dxdy = \frac{hk}{4}$[(Sum of 4 corner values)+ 2(Sum of other boundary values) + 4(Sum of interior values)]
where $h = \frac{b-a}{m}, k = \frac{d-c}{n}$.

541. **Double integral using Simpson's rule:**(3x3 grid)
$\int_c^d\int_a^b f(x, y)dxdy = \frac{hk}{9}$[(Sum of 4 corner values) + 4(Sum of other boundary values) + 16(interior value)] where $h = \frac{b-a}{2}, k = \frac{d-c}{2}$.

542. **Taylor series method:** (First order IVP)
Solution of $y' = f(x, y), y(x_0) = y_0$ is given by $y_1 = y(x_0 + h) = y_0 + hy'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$ where $y'_0 = \left[\frac{dy}{dx}\right]_{(x_0,y_0)}$; $y''_0 = \left[\frac{d^2y}{dx^2}\right]_{(x_0,y_0)}$; $y'''_0 = \left[\frac{d^3y}{dx^3}\right]_{(x_0,y_0)}$.

543. **Euler's method:**
Solution of $y' = f(x, y), y(x_0) = y_0$ is given by $y_1 = y(x_0 + h) = y_0 + hf(x_0, y_0)$.

544. **Modified Euler's method:**
Solution of $y' = f(x, y)$ with $y(x_0) = y_0$ is given by $y_1 = y(x_0 + h) = y_0 + hf(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0))$.

545. **Fourth order Runge-Kutta method (RK-IV Method)** (First order IVP)
Solution of $y' = f(x, y)$ with $y(x_0) = y_0$ is given by $y_1 = y(x_0 + h) = y_0 + \Delta y$ where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k4)$; $k_1 = hf(x_0, y_0); k_2 = hf(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}); k_3 = hf(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}); k_4 = hf(x_0+h, y_0+k_3)$.

546. **Fourth order Runge-Kutta method (RK-IV Method)** (First order simultaneous IVP)
Solution of $y' = f(x, y, z), z' = g(x, y, z)$ with $y(x_0) = y_0, z(x_0) = z_0$ is given by $y_1 = y(x_0 + h) = y_0 + \Delta y$ and $z_1 = z(x_0 + h) = z_0 + \Delta z$ where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); k_1 = hf(x_0, y_0, z_0); k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}); k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}); k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$ and $\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4); l_1 = hg(x_0, y_0, z_0); l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}); l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}); l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$.

547. **Fourth order Runge-Kutta method (RK-IV Method)** (Second order IVP)
To solve $y'' = \phi(x, y, y')$ with $y(x_0) = y_0, y'(x_0) = y'_0$, put $y' = z$ then, we have $y' = z, z' = \phi(x, y, z)$ with $y(x_0) = y_0, z(x_0) = z_0$. Now apply RK-IV method for simultaneous equations.

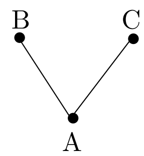
548. **Milne's Predictor-Corrector method:**
To solve $y' = f(x, y)$ with $y(x_{n-3}) = y_{n-3}$, $y(x_{n-2}) = y_{n-2}, y(x_{n-1}) = y_{n-1}, y(x_n) = y_n$, the predictor formula is
$y_{n+1,P} = y_{n-3}+\frac{4h}{3}[2y'_{n-2}-y'_{n-1}+2y'_n]$ and the corrector formula is
$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$.

549. **Finite difference approximations for derivatives:**
$y'_i = \frac{y_{i+1}-y_{i-1}}{2h}$ and $y''_i = \frac{y_{i-1}-2y_i+y_{i+1}}{h^2}$.

550. **Classification of PDE:** The second order linear PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where A, B, C, D, E, F, G are functions of x and y, is said to be (i) elliptic if $B^2-4AC < 0$; (ii) parabolic if $B^2-4AC = 0$; (iii) hyperbolic if $B^2 - 4AC > 0$.

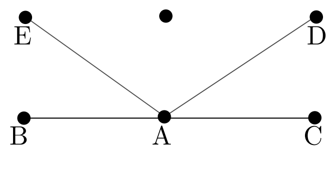
551. **Finite difference approximations for partial derivatives:**
We divide the xy-plane into a network of rectangles of sides h and k by drawing lines $x = ih$ and $y = jk$ for $i, j = 0, 1, 2, \cdots$. The points of intersection of these family of lines are called *mesh points* or *grid points.* Let $u(x, y) = u(ih, jk) = u_{i,j}$. We have
$u_x = \frac{u_{i+1,j}-u_{i,j}}{h}$ (forward difference);
$u_x = \frac{u_{i,j}-u_{i-1,j}}{h}$ (backward difference);
$u_x = \frac{u_{i+1,j}-u_{i-1,j}}{2h}$ (central difference);
$u_y = \frac{u_{i,j+1}-u_{i,j}}{k}$ (forward difference);
$u_y = \frac{u_{i,j}-u_{i,j-1}}{k}$ (backward difference);
$u_y = \frac{u_{i,j+1}-u_{i,j-1}}{2k}$ (central difference);
Also $u_{xx} = \frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}$ and $u_{yy} = \frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{k^2}$.

552. **Bender-Schmidt Method**(Explicit Method): To solve $u_{xx} - au_t = 0$ with the boundary conditions $u(0, t) = T_0, u(l, t) = T_l$ and the initial condition $u(x, 0) = f(x), 0 < x < l$.
Explicit formula: $u_{i,j+1} = \lambda u_{i+1,j} + (1-2\lambda)u_{i,j} + \lambda u_{i-1,j}$ where $\lambda = \frac{k}{ah^2}$.
Bender-Schmidt recurrence formula or simple scheme:(when $\lambda = \frac{1}{2}$ or $k = \frac{ah^2}{2}$).
$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$.



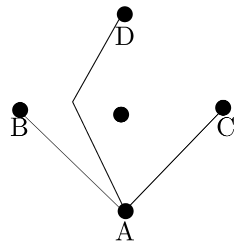
i.e., Value of u at $A = \frac{1}{2}$[Value of u at B + Value u at C].

553. **Crank-Nicholson's Simple Scheme**(Implicit Method): To solve $u_{xx} - au_t = 0$ with the boundary conditions $u(0, t) = T_0, u(l, t) = T_l$ and the initial condition $u(x, 0) = f(x), 0 < x < l$, the Crank-Nicholson's simple scheme is
$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$ when $k = ah^2$.



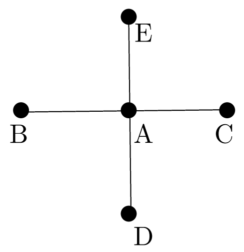
i.e., Value of u at A = Average value of u at B, C, D and E.

554. To solve $a^2uxx - u_{tt} = 0$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and the initial conditions $u(x,0) = f(x), u_t(x,0) = 0$, the explicit formula is
$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ when $k = \frac{h}{a}$.



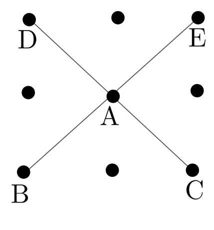
i.e., Value of u at A = Value of u at B + Value of u at C - Value of u at D.

555. To solve the Laplace equation $\nabla^2 u = 0$, the Standard five point formula(SFPF) is
$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$ when $h = k = 1$.



i.e., Value of u at A = Average value of u at B, C, D and E.

556. The diagonal five point formula (DFPF) to solve the Laplace equation is
$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}]$.



i.e., Value of u at A = Average value of u at B, C, D and E.

557. To solve $\nabla^2 u = f(x,y)$, apply the following finite difference formula
$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$
at each grid point, we get a system of linear equations which can be solved easily.

23 Bessel Functions

558. Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

559.

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} k!(n+k)!}$$

560.

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} x^{2k+n}}{2^{2k+n} k!(n+k)!}$$

561. $J_{-n}(x) = (-1)^n J_n(x)$

562. $J_0(0) = 1, J_n(0) = 0, n = 1, 2, 3, \cdots$

563. $xJ_n'(x) = xJ_{n-1}(x) - nJ_n(x)$.

24 Legendre Polynomial

564. Legendre's differential equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$$

where n is a real constant.Equivalently,

$$\{(1 - x^2)y'\}' + n(n+1)y = 0.$$

565. Rodrigue's formula for Legendre's polynomial:

$$P_n(x) = \frac{1}{n!2^n}\frac{d^n}{dx^n}(x^2 - 1)^n.$$

566. $P_0(x) = 1; P_1(x) = x; P_2(x) = \frac{1}{2}(3x^2 - 1)$.

567. $P_3(x) = \frac{1}{2}(5x^3 - 3x)$.

568. $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$.

569. Generating function for Legendre's polynomial. For $t \neq 1$

$$\sum_{n=0}^{\infty} P_n(x)t^n = \frac{1}{\sqrt{1 - 2xt + t^2}}.$$

570. Bonnet's recurrence relation
$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$.

571. $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$.

572. $P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$.

573. Orthogonal property:

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

25 Partitions

574. Shifted factorial:
$(a)_0 = 1; (a)_n = a(a+1)(a+2)\cdots(a+n-1)$

575. Gaussian Hypergeometric equation:
$x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0$ has singular points $x=0, x=1$ and its solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

q-shifted factorial:

576. $(a;q)_0 = 1$

577. $(a;q)_n = \prod_{i=0}^{n-1}(1-aq^i)$
$= (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$

578. $(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i)$
$= (1-a)(1-aq)(1-aq^2)\cdots$

579. $(a;q)_n = (a;q)_\infty \div (aq^n;q)_\infty$

580. $(a^2;q^2)_n = (a;q)_n(-a;q)_n$

581. $(a^2;q^2)_\infty = (a;q)_\infty(-a;q)_\infty$

582. $(a;q^2)_n(aq;q^2)_n = (a;q)_{2n}$

583. $\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n$

584. q-binomial therorem (Cauchy's theorem): If $|q|<1, |t|<1$ then

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \prod_{n=0}^{\infty} \frac{(1-atq^n)}{(1-tq^n)} = \frac{(at;q)_\infty}{(t;q)_\infty}$$

585. Heine's Transformation:
For $|q|<1, |t|<1, |b|<1$, $\sum_{n=0}^{\infty} \frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n} t^n = \frac{(b:q)_\infty(at;q)_\infty}{(c:q)_\infty(t;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m(t;q)_m}{(at;q)_m(q;q)_m} b^m$

586. A partition of a positive integer n, is the representation of n as a sum of positive integers, called summands or parts.

587. $p(n)$ - Number of partitions of n

588. $c(n)$ - Number of ordered partitions or compositions of n

589. $p_m(n)$ - Number of partitions of n with parts $\leq m$

590. $D(n)$ - Number of partitions of n into distinct parts.

591. $p(S,m,n)$ - Number of partitions of n into m parts each taken from the set S, a subset of positive integers.

592. $p_d(S,m,n)$ - Number of partitions of n into m parts each taken from the set S, a subset of positive integers, and have distinct parts.

593. $O(n)$ - Number of partitions of n into odd parts.

594. $c(m,n)$ - Number of compositions of n into m parts $= \binom{n-1}{m-1}$

595. $c_{m,n}(s)$ - Number of compositions of n into m parts, each part is $\leq s$.

596. Generating function for $p_m(n)$:

$$\sum_{n=0}^{\infty} p_m(n) q^n = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$

597. Generating funciton for $D(n)$:

$$\sum_{n=0}^{\infty} D(n) q^n = \prod_{n=1}^{\infty}(1+q^n)$$

598. Generating function for $p(S,m,n)$:

$$\sum_{n\geq 0}\sum_{m\geq 0} p(S,m,n) z^m q^n = \prod_{n\in S} \frac{1}{1-zq^n}$$

599. Generating function for $p_d(S,m,n)$:

$$\sum_{n\geq 0}\sum_{m\geq 0} p_d(S,m,n) z^m q^n = \prod_{n\in S}(1+zq^n)$$

600. Generating function for $O(n)$:

$$\sum_{n=0}^{\infty} O(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^{2n-1}}$$

601. $\lim_{m\to\infty} p_m(n) = p(n)$

602. $c(n) = 2^{n-1}$

603. Jacobi's Triple product Identity(JTP):
For $z \neq 0, |q|<1$,

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (q^2;q^2)_\infty(-zq;q^2)_\infty(-z^{-1}q;q^2)_\infty$$

$$= \prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})(1+z^{-1}q^{2n+1})$$

604. If $|t| < 1, |q| < 1$,

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \prod_{n=0}^{\infty} (1 - tq^n)^{-1} = \frac{1}{(t;q)_\infty}$$

605. If $|t| < 1, |q| < 1$,

$$\sum_{n=0}^{\infty} \frac{t^n q^{\binom{n}{2}}}{(q;q)_n} = \prod_{n=0}^{\infty} (1 + tq^n) = (-t;q)_\infty$$

606.

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty}$$

607. $c(n) = \sum_{m=1}^{n} c(m,n) = 2^{n-1}$

608. q - Jacobi Polynomial or Gaussian polynomial

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \ 0 \le k \le n$$

called the q- Jabobi binomila coefficient or Gaussian polynomial.

609. $\lim_{q \to 1} \begin{bmatrix} n \\ k \end{bmatrix} = \binom{n}{k}$

610. $p(N, M, n)$ - Number of partitions of n into atmost M parts, each part $\le N$.

611. $p(N, M, n) = \begin{cases} 0 & \text{if } n > NM \\ 1 & \text{if } n = NM \\ 1 & \text{if } N = 0 \text{ or } M = 0 \end{cases}$

612. Generating funciton for $p(N, M, n)$ is

$$\sum_{n=0}^{MN} p(N, M, n) q^n = \frac{(q;q)_{N+M}}{(q;q)_N (q;q)_M} = \begin{bmatrix} N + M \\ M \end{bmatrix}$$

26 Infinite Products

613. $\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^{k+1}}{2k-1}\right) = \sqrt{2}$

614. $\prod_{k=2}^{\infty} \left(1 - \frac{1}{k^2}\right) = \frac{1}{2}$

615. $\prod_{k=1}^{\infty} \left(1 - \frac{1}{(2k+1)^2}\right) = \frac{\pi}{4}$

27 Number Theory

616. $a \mid b$ imply $a \mid bc$ for any integer c.

617. $a \mid b$ and $b \mid c$ imply $a \mid c$.

618. $a \mid b$ and $a \mid c$ imply $a \mid (bx + cy)$ for any integers x and y.

619. $a \mid b$ and $b \mid a$ imply $a = \pm b$.

620. For any positive integer m, $(ma, mb) = m(a, b)$.

621. $a \equiv b \pmod{m}, b \equiv a \pmod{m}$, and $a - b \equiv 0 \pmod{m}$ are equivalent statements.

622. $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$ then $a \equiv c \pmod{m}$.

623. $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ax + by \equiv cx + dy \pmod{m}$.

624. $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ then $ac \equiv bd \pmod{m}$.

625. $a \equiv b \pmod{m}$ and $d|m, d > 0$, then $a \equiv b \pmod{d}$.

626. The number $\phi(m)$ is the number of positive integers less than or equal to m that are relatively prime to m.

627. (Fermat's theorem) Let p denote a prime. If $p \nmid a$ then $a^{p-1} \equiv 1 \pmod{p}$. For every integer a, $a^p \equiv a \pmod{p}$.

628. (Euler's generalization of Fermat's theorem) If $(a, m) = 1$ then $a^{\phi(m)} \equiv 1 \pmod{m}$.

629. (Wilson's theorem) If p is a prime then $(p-1)! \equiv -1 \pmod{p}$.

630. Let m and n denote any two positive, relatively prime integers. Then $\phi(mn) = \phi(m)\phi(n)$.

631. If $n > 1$ then $\phi(n) = n \prod_{p|n} (1 - \frac{1}{p})$. Also $\phi(1) = 1$.

632. For $n \ge 1$ we have $\sum_{d|n} \phi(d) = n$