MAT211 - Numerical Methods : Important Formulae

1 Solution of Equations and Eigen value Problems

1. **Intermediate Theorem:** Let $f(x)$ be a continuous function on $[a, b]$. If $d \in [f(a), f(b)]$ then there exists a $c \in [a, b]$ such that $f(c) = d$.

2. $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where a_i's are constants, is called an *Algebraic equation* or a *Polynomial equation* of n^{th} degree, if $a_0 \neq 0$. A non-algerbraic equation is called a *Transcendental equation.*

3. If $f(x) = 0$ is continuous in $a \leq x \leq b$ and if $f(a)$ and $f(b)$ are of opposite signs, the $f(\xi) = 0$ for at least one number ξ such that $a < \xi < b$.

4. Every algebraic equation of odd degree has at least one real root whose sign is opposite to that of its last term.

5. Every algebraic equation of even degree with last term negative, has at least a pair of real roots one is positive and the other negative.

6. **Iterative Method :** To find a real root of $f(x) = 0$ by iterative method, rewrite the given equation in the form of $x = \phi(x)$ where $|\phi'(x)| < 1$ in (a, b). (Note: $f(a)$ and $f(b)$ are of opposite signs).

7. **Regula-Falsi Method:** To find a real root of $f(x) = 0$, find two real numbers a and b such that $f(a)$ and $f(b)$ are of opposite signs. The first approximate root is given by $x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)}$.

8. **Newton-Raphson Method**(single variable): To find a real root of $f(x) = 0$ by Newton-Raphson method, the first approximate root is given by $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$ where x_0 is an initial approximate root of $f(x) = 0$. In general $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$. (Note: Choose $x_0 = \frac{a+b}{2}$ if $f(a)$ and $f(b)$ are of opposite signs).

9. The order of convergence of Newton-Raphson method is two.

10. The condition for convergence of Newton-Raphson method is $|f(x)f''(x)| < |f'(x)|^2$.

11. **Newton-Raphson Method**(Two variables):Consider the system of equations $f(x, y) = 0$ and $g(x, y) = 0$ with the initial approximation being (x_0, y_0). The first approximate root is given by $x_1 = x_0 + h, y_1 = y_0 + k$ where $h = -\frac{D_1}{D}, k = -\frac{D_2}{D}$,
$D = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}_{(x_0,y_0)}, D_1 = \begin{vmatrix} f & f_y \\ g & g_y \end{vmatrix}_{(x_0,y_0)}, D_2 = \begin{vmatrix} f_x & f \\ g_x & g \end{vmatrix}_{(x_0,y_0)}$

12. **Gauss Elimination Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss Elimination method*, transform the augmented matrix $[A, B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ into an upper triangular matrix by the elementary row operations. Back substitution on the transformed equations will give the required solution.

13. **Gauss-Jordan Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss-Jordan method*, transform the augmented matrix $[A, B] = \begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}$ into a diagonal matrix by the elementary row operations. The transformed equations will give the required solution.

14. A square matrix is said to be *diagonally dominant* if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical values of other elements in that row.

15. **Gauss-Jacobi Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss-Jordan method*. Assume that the coefficient matrix is diagonally dominant. Rewrite the given system of equations as $x = \frac{1}{a_1}(d_1 - b_1y - c_1z); y = \frac{1}{b_2}(d_2 - a_2x - c_2z); z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$. If $x^{(0)}, y^{(0)}, z^{(0)}$ be the initial guess for x, y, z respectively, then the first approximate root is given by $x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}); y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}); z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)})$. Using these values, we get the second approximate root as $x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)}); y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(1)}); z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$. Repeating this process, we get the required solution.

16. **Gauss-Seidel Method:** To solve the system of linear equations $a_1x + b_1y + c_1z = d_1, a_2x + b_2y + c_2z = d_2, a_3x + b_3y + c_3z = d_3$ by *Gauss-Seidel method*. Assume that the coefficient matrix is diagonally dominant. Rewrite the given system of equations as $x = \frac{1}{a_1}(d_1 - b_1y - c_1z); y = \frac{1}{b_2}(d_2 - a_2x - c_2z); z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$. If $y^{(0)}, z^{(0)}$ be the initial guess for y, z respectively, then the first approximate root is given by $x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)}); y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)}); z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$. Using these values, we get the second approximate root as $x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)}); y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(2)} - c_2z^{(1)}); z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(2)} - b_3y^{(2)})$. Repeating this process, we get the required solution.

17. **Power Method:** To find the numerically largest eigen value and the corresponding eigen vector of a square matrix A by Power method, choose the initial eigen vector as $X_0 = (1, 0, 0)'$.
Then
$AX_0 = \lambda_1 X_1$ where $X_1 = (1, a_1, b_1)'$
$AX_1 = \lambda_2 X_2$ where $X_2 = (1, a_2, b_2)'$
$AX_2 = \lambda_3 X_3$ where $X_3 = (1, a_3, b_3)'$
The sequence $\lambda_1, \lambda_2, \lambda_3, \cdots$ converges to the numerically largest eigen value of A and the sequence $X_1, X_2, X_3, \cdots$ converges to the corresponding eigen vector.

2 Interpolation

Interpolation with equal intervals

18. **Finite Differences:** Let $y = f(x)$ be a given function of x and let $y_i = f(x_i)$ for $i = 0, 1, 2, \cdots, n$ where $x_i = x_{i-1} + h$ for $i = 1, 2, \cdots, n$, h is the interval of differencing. Now $y_1 - y_0, y_2 - y_1, \cdots, y_n - y_{n-1}$ are called the first differences of the function y. We denote $\Delta y_i = y_{i+1} - y_i$, Δ is called the forward difference operator. Now the differences of these first differences are called second differences. Thus $\Delta^2 y_i = \Delta(\Delta y_i) = \Delta(y_{i+1} - y_i) = \Delta y_{i+1} - \Delta y_i = (y_{i+2} - y_{i+1}) - (y_{i+1} - y_i) = y_{i+2} - 2y_{i+1} + y_i$. In general, $\Delta^k y_i = \Delta^{k-1} y_{i+1} - \Delta^{k-1} y_i$.

19. $\Delta f(x) = f(a + h) - f(x)$.

20. **Forward difference table**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

21. **Newton's Forward Interpolation Formula**
$y_n = y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$
(This formula gives greater accuracy when $x_0 + nh$ is near the beginning of the table)

22. **Backward Difference:** The backward difference operator ∇ is defined as $\nabla y_i = y_i - y_{i-1}$. The second backward difference is $\nabla^2 y_i = \nabla(\nabla y_i) = \nabla(y_i - y_{i-1}) = \nabla y_i - \nabla y_{i-1} = (y_i - y_{i-1}) - (y_{i-1} - y_{i-2}) = y_i - 2y_{i-1} + y_{i-2}$. In general $\nabla^k y_i = \nabla^{k-1} y_i - \nabla^{k-1} y_{i-1}$.

23. $\nabla f(x) = f(x) - f(x-h)$.

24. **Backward difference table**

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_{-4}	y_{-4}				
		∇y_{-3}			
x_{-3}	y_{-3}		$\nabla^2 y_{-2}$		
		∇y_{-2}		$\nabla^3 y_{-1}$	
x_{-2}	y_{-2}		$\nabla^2 y_{-1}$		$\nabla^4 y_0$
		∇y_{-1}		$\nabla^3 y_0$	
x_{-1}	y_{-1}		$\nabla^2 y_0$		
		∇y_0			
x_0	y_0				

25. **Newton's Backward Interpolation Formula**
$y_n = y(x_0 + nh) = y_0 + n\nabla y_0 + \frac{n(n+1)}{2!}\nabla^2 y_0 + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_0 + \cdots$
(This formula gives greater accuracy when $x_0 + nh$ is near the end of the table)

26. **Central difference:** The central difference operator δ is defined as $\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$.

27. $\delta y_x = \Delta y_{x-\frac{h}{2}}$ or $\Delta y_x = \delta y_{x+\frac{h}{2}}$.

28. **Central difference table**

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_{-2}	y_{-2}				
		Δy_{-2}			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		
		$\mathbf{\Delta y_{-1}}$		$\mathbf{\Delta^3 y_{-2}}$	
$\mathbf{x_0}$	$\mathbf{y_0}$		$\mathbf{\Delta^2 y_{-1}}$		$\mathbf{\Delta^4 y_{-2}}$
		$\mathbf{\Delta y_0}$		$\mathbf{\Delta^3 y_{-1}}$	
x_1	y_1		$\Delta^2 y_0$		
		Δy_1			
x_2	y_2				

29. **Stirling's Central difference formula**
$y(x_0+nh) = y_0 + n\left(\frac{\Delta y_0+\Delta y_{-1}}{2}\right) + \frac{n^2}{2!}\Delta^2 y_{-1} + \frac{n(n^2-1)}{3!}\left(\frac{\Delta^3 y_{-1}+\Delta^3 y_{-2}}{2}\right) + \frac{n^2(n^2-1)}{4!}\Delta^4 y_{-2} + \cdots$

Interpolation with unequal intervals

30. **Divided Differences:** Let $y = f(x)$ be the given function. Let $f(x_0), f(x_1), \cdots, f(x_n)$ be the values of the function corresponding to the arguments $x_0, x_1, \cdots, x_n$ and $x_0, x_1, \cdots, x_n$ need not be equally spaced. The first divided difference of $f(x)$ for the arguments x_0 and x_1 is defined as $f[x_0, x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$. Similarly, $f[x_1, x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. The second divided difference of $f(x)$ for the arguments x_0, x_1 and x_2 is $f[x_0, x_1, x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$.

31. **Divided difference table**

x	$f(x)$	First divided difference	Second divided difference	Third divided difference	Fourth divided difference
x_0	$f(x_0)$				
		$f[x_0,x_1] = \frac{f(x_1)-f(x_0)}{x_1-x_0}$			
x_1	$f(x_1)$		$f[x_0,x_1,x_2]^*$		
		$f[x_1,x_2] = \frac{f(x_2)-f(x_1)}{x_2-x_1}$		$f[x_0,x_1,x_2,x_3]^*$	
x_2	$f(x_2)$		$f[x_1,x_2,x_3]^*$		$f[x_0,x_1,x_2,x_3,x_4]^*$
		$f[x_2,x_3] = \frac{f(x_3)-f(x_2)}{x_3-x_2}$		$f[x_1,x_2,x_3,x_4]^*$	
x_3	$f(x_3)$		$f[x_2,x_3,x_4]^*$		
		$f[x_3,x_4] = \frac{f(x_4)-f(x_3)}{x_4-x_3}$			
x_4	$f(x_4)$				

$^*f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$; $f[x_1,x_2,x_3] = \frac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$; $f[x_2,x_3,x_4] = \frac{f[x_3,x_4]-f[x_2,x_3]}{x_4-x_2}$;
$f[x_0,x_1,x_2,x_3] = \frac{f[x_1,x_2,x_3]-f[x_0,x_1,x_2]}{x_3-x_0}$; $f[x_1,x_2,x_3,x_4] = \frac{f[x_2,x_3,x_4]-f[x_1,x_2,x_3]}{x_4-x_1}$;
$f[x_0,x_1,x_2,x_3,x_4] = \frac{f[x_1,x_2,x_3,x_4]-f[x_0,x_1,x_2,x_3]}{x_4-x_0}$.

32. **Newton's Divided difference formula:**
$f(x) = f(x_0) + (x-x_0)f[x_0,x_1] + (x-x_0)(x-x_1)f[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)f[x_0,x_1,x_2,x_3] + (x-x_0)(x-x_1)(x-x_2)(x-x_3)f[x_0,x_1,x_2,x_3,x_4] + \cdots$

33. **Lagranges Interpolation formula:** Let $y = f(x)$ be the given function.
Let $f(x_0), f(x_1), \cdots, f(x_n)$ be the values of the function corresponding to the arguments $x_0, x_1, \cdots, x_n$ and $x_0, x_1, \cdots, x_n$ need not be equally spaced.
$f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}f(x_1) + \cdots$
$+ \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$ is called the Lagrange's interpolation formula.

3 Numerical Differentiation and Integration:

34. **Numerical Differentiation:** Find the polynomial from the given data using Newton's Forward/divided difference interpolation formula and hence find its derivative at the desired point.

35. Newton's forward interpolation formula to find the derivative at $x = x_0$ is
$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \cdots\right]$ and $f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$

36. Newton's backward interpolation formula to find the derivative at $x = x_0$ is
$f'(x_0) = \frac{1}{h}\left[\nabla y_0 + \frac{\nabla^2 y_0}{2} + \frac{\nabla^3 y_0}{3} + \frac{\nabla^4 y_0}{4} + \cdots\right]$ and $f''(x_0) = \frac{1}{h^2}\left[\nabla^2 y_0 + \nabla^3 y_0 + \frac{11}{12}\nabla^4 y_0 + \cdots\right]$

37. **Trapezoidal rule:**
$\int_a^b f(x)dx = \frac{h}{2}\left[(y_0+y_n) + 2(y_1+y_2+\cdots+y_{n-1})\right]$ where
$h = \frac{b-a}{n}$, n- Number of intervals; $x_0 = a$; $x_n = b$; $x_i = x_{i-1} + h$; $y_i = f(x_i)$

38. **Simpson's $\frac{1}{3}^{rd}$ rule:**
$\int_a^b f(x)dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots)+4(y_1+y_3+\cdots)\right]$ where
$h=\frac{b-a}{n}$, n- Number of intervals(even); $x_0=a;\ x_n=b;\ x_i=x_{i-1}+h;\ y_i=f(x_i)$

39. **Simpson's $\frac{3}{8}^{th}$ rule:**
$\int_a^b f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots)+2(y_3+y_6+\cdots)\right]$ where
$h=\frac{b-a}{n}$, n- Number of intervals(a multiple of 3); $x_0=a;\ x_n=b;\ x_i=x_{i-1}+h;\ y_i=f(x_i)$

40. **Gaussian two point quadrature formula:**
$\int_{-1}^{1} f(x)dx = f(-\frac{1}{\sqrt{3}})+f(\frac{1}{\sqrt{3}})$

41. **Gaussian three point quadrature formula:**
$\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$

42. To evaluate $\int_a^b f(x)dx$ by Gaussian quadrature formula, put $t=\frac{(b-a)x+(b+a)}{2}$ then apply Gaussian quadrature formula.

43. **Double integral using Trapezoidal rule:**
$\int_c^d\int_a^b f(x,y)dxdy = \frac{hk}{4}$[(Sum of 4 corner values) + 2(Sum of other boundary values) + 4(Sum of interior values)] where $h=\frac{b-a}{m}, k=\frac{d-c}{n}$.

44. **Double integral using Simpson's rule:**(3x3 grid)
$\int_c^d\int_a^b f(x,y)dxdy = \frac{hk}{9}$[(Sum of 4 corner values) + 4(Sum of other boundary values) + 16(interior value)] where $h=\frac{b-a}{2}, k=\frac{d-c}{2}$.

4 Initial Value Problems:

Single Step Methods:

45. **Taylor series method:** (First order IVP)
Solution of $y'=f(x,y), y(x_0)=y_0$ is given by
$y_1=y(x_0+h)=y_0+hy_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\cdots$
where $y_0'=\left[\frac{dy}{dx}\right]_{(x_0,y_0)}; y_0''=\left[\frac{d^2y}{dx^2}\right]_{(x_0,y_0)}; y_0'''=\left[\frac{d^3y}{dx^3}\right]_{(x_0,y_0)}$.

46. **Euler's method:**
Solution of $y'=f(x,y), y(x_0)=y_0$ is given by
$y_1=y(x_0+h)=y_0+hf(x_0,y_0)$.

47. **Modified Euler's method:**
Solution of $y'=f(x,y)$ with $y(x_0)=y_0$ is given by
$y_1=y(x_0+h)=y_0+hf(x_0+\frac{h}{2},y_0+\frac{h}{2}f(x_0,y_0))$.

48. **Fourth order Runge-Kutta method (RK-IV Method)** (First order IVP)
Solution of $y'=f(x,y)$ with $y(x_0)=y_0$ is given by $y_1=y(x_0+h)=y_0+\Delta y$
where $\Delta y=\frac{1}{6}(k_1+2k_2+2k_3+k4)$;
$k_1=hf(x_0,y_0); k_2=hf(x_0+\frac{h}{2},y_0+\frac{k_1}{2}); k_3=hf(x_0+\frac{h}{2},y_0+\frac{k_2}{2}); k_4=hf(x_0+h,y_0+k_3)$.

49. **Fourth order Runge-Kutta method (RK-IV Method)**
(First order simultaneous IVP)
Solution of $y' = f(x, y, z), z' = g(x, y, z)$ with $y(x_0) = y_0, z(x_0) = z_0$ is given by
$y_1 = y(x_0 + h) = y_0 + \Delta y$ and $z_1 = z(x_0 + h) = z_0 + \Delta z$
where $\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4); k_1 = hf(x_0, y_0, z_0); k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2});$
$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}); k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$ and
$\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4); l_1 = hg(x_0, y_0, z_0); l_2 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2});$
$l_3 = hg(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}); l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$.

50. **Fourth order Runge-Kutta method (RK-IV Method)** (Second order IVP)
To solve $y'' = \phi(x, y, y')$ with $y(x_0) = y_0, y'(x_0) = y'_0$, put $y' = z$ then, we have $y' = z, z' = \phi(x, y, z)$ with $y(x_0) = y_0, z(x_0) = z_0$. Now apply RK-IV method for simultaneous equations.

Multistep Method

51. **Milne's Predictor-Corrector method:** To solve $y' = f(x, y)$ with
$y(x_{n-3}) = y_{n-3}, y(x_{n-2}) = y_{n-2}, y(x_{n-1}) = y_{n-1}, y(x_n) = y_n$, the predictor formula is
$y_{n+1,P} = y_{n-3} + \frac{4h}{3}[2y'_{n-2} - y'_{n-1} + 2y'_n]$ and the corrector formula is
$y_{n+1,C} = y_{n-1} + \frac{h}{3}[y'_{n-1} + 4y'_n + y'_{n+1}]$.

5 Boundary Value Problems

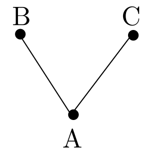
52. **Finite difference approximations for derivatives:**
$y'_i = \frac{y_{i+1} - y_{i-1}}{2h}$ and $y''_i = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2}$.

53. **Classification of PDE:** The second order linear PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, where A, B, C, D, E, F, G are functions of x and y, is said to be (i) elliptic if $B^2 - 4AC < 0$; (ii) parabolic if $B^2 - 4AC = 0$; (iii) hyperbolic if $B^2 - 4AC > 0$.

54. **Finite difference approximations for partial derivatives:**
We divide the xy-plane into a network of rectangles of sides h and k by drawing lines $x = ih$ and $y = jk$ for $i, j = 0, 1, 2, \cdots$. The points of intersection of these family of lines are called *mesh points* or *grid points*. Let $u(x, y) = u(ih, jk) = u_{i,j}$. We have $u_x = \frac{u_{i+1,j} - u_{i,j}}{h}$ (forward difference); $u_x = \frac{u_{i,j} - u_{i-1,j}}{h}$ (backward difference); $u_x = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$ (central difference); $u_y = \frac{u_{i,j+1} - u_{i,j}}{k}$ (forward difference); $u_y = \frac{u_{i,j} - u_{i,j-1}}{k}$ (backward difference); $u_y = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$ (central difference); Also $u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$ and $u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$.

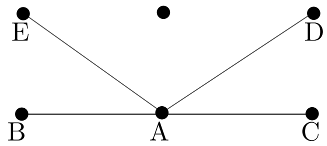
Solution of one dimensional Heat flow Equation

55. **Bender-Schmidt Method**(Explicit Method): To solve $u_{xx} - au_t = 0$ with the boundary conditions $u(0, t) = T_0, u(l, t) = T_l$ and the initial condition $u(x, 0) = f(x), 0 < x < l$.
Explicit formula: $u_{i,j+1} = \lambda u_{i+1,j} + (1 - 2\lambda)u_{i,j} + \lambda u_{i-1,j}$ where $\lambda = \frac{k}{ah^2}$.
Bender-Schmidt recurrence formula or simple scheme:(when $\lambda = \frac{1}{2}$ or $k = \frac{ah^2}{2}$).
$u_{i,j+1} = \frac{1}{2}[u_{i-1,j} + u_{i+1,j}]$.



i.e., Value of u at $A = \frac{1}{2}$[Value of u at B + Value u at C].

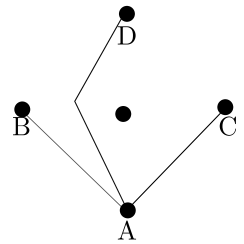
56. **Crank-Nicholson's Simple Scheme**(Implicit Method): To solve $u_{xx} - au_t = 0$ with the boundary conditions $u(0, t) = T_0, u(l, t) = T_l$ and the initial condition $u(x, 0) = f(x), 0 < x < l$, the Crank-Nicholson's simple scheme is
$u_{i,j+1} = \frac{1}{4}[u_{i-1,j+1} + u_{i+1,j+1} + u_{i-1,j} + u_{i+1,j}]$ when $k = ah^2$.



i.e., Value of u at A = Average value of u at B, C, D and E.

Solution of one dimensional wave equation

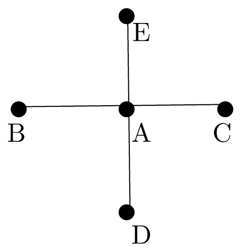
57. To solve $a^2uxx - u_{tt} = 0$ with the boundary conditions $u(0,t) = u(l,t) = 0$ and the initial conditions $u(x,0) = f(x), u_t(x,0) = 0$, the explicit formula is
$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$ when $k = \frac{h}{a}$.



i.e., Value of u at A = Value of u at B + Value of u at C - Value of u at D.

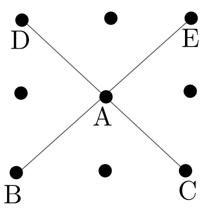
Solution of Laplace Equation

58. To solve the Laplace equation $\nabla^2 u = 0$, the Standard five point formula(SFPF) is
$u_{i,j} = \frac{1}{4}[u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1}]$ when $h = k = 1$.



i.e., Value of u at A = Average value of u at B, C, D and E.

59. The diagonal five point formula (DFPF) to solve the Laplace equation is
$u_{i,j} = \frac{1}{4}[u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1}]$.



i.e., Value of u at A = Average value of u at B, C, D and E.

Solution of Poisson Equation

60. To solve $\nabla^2 u = f(x,y)$, apply the following finite difference formula
$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = h^2 f(ih, jh)$
at each grid point, we get a system of linear equations which can be solved easily.