Dr. K. Karuppasamy

www.drkk.in

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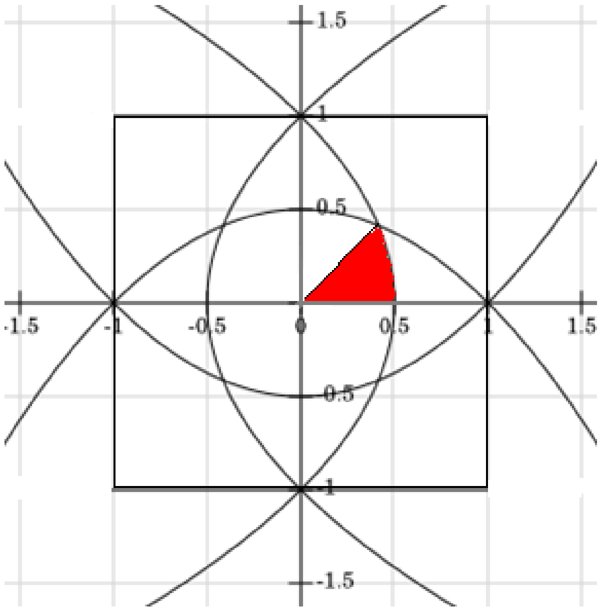
Problem: Find the area of the region consists of all the points inside a square of 2 cm by 2 cm such that they are closer to the center of the square than its sides.

Solution: Let the square be bounded by the lines x = 1, x = -1, y = 1 and y = -1.

Let O be the origin and P(x,y) be any point on the first quadrant such that P is closer to the origin than the sides x = 1 and y = 1.



Thus we have, $x^2+y^2 < (x-1)^2$ and $x^2+y^2 < (y-1)^2$

⇨ $y^2+2x-1 < 0$ and $x^2+2y-1 < 0$

Solving $y^2+2x-1 = 0$ and $x^2+2y-1 = 0$, we get the point of intersection in the first quadrant as

$\left(-1+\sqrt{2}, -1+\sqrt{2}\right)$

Put $x = r\cos\theta, y = r\sin\theta$ in $y^2+2x-1 = 0$, we get

$r = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)$. In the red shaded region, θ varies from 0 to $\frac{\pi}{4}$ and r varies from 0 to $\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)$.

By symmetry, we have

Area of the required region = 8 (Area of the red shaded region)

$$= 8\int_0^{\frac{\pi}{4}}\int_0^{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)} r\,dr\,d\theta = 8\int_0^{\frac{\pi}{4}}\left[\frac{r^2}{2}\right]_0^{\frac{1}{2}\sec^2\left(\frac{\theta}{2}\right)} d\theta = \int_0^{\frac{\pi}{4}}\sec^4\left(\frac{\theta}{2}\right)d\theta = \int_0^{\frac{\pi}{4}}\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)\sec^2\left(\frac{\theta}{2}\right)d\theta$$

$$= 2\int_0^{\sqrt{2}-1}\left(1+u^2\right)du \quad \text{(Put } u = \tan\left(\frac{\theta}{2}\right)\text{)}$$

$$= 2\left[u+\frac{u^3}{3}\right]_0^{\sqrt{2}-1} = \frac{2}{3}\left[3\left(\sqrt{2}-1\right)+\left(\sqrt{2}-1\right)^3\right] = \frac{2}{3}\left(\sqrt{2}-1\right)\left[3+\left(\sqrt{2}-1\right)^2\right] = \frac{4}{3}\left(4\sqrt{2}-5\right).$$