

Yahoo Answer dated 02-12-2013

**Question:** (a) Show that  $\int_0^\pi xf(\sin x)dx = \frac{\pi}{2} \int_0^\pi f(\sin x)dx$   
 (b) Use part (a) to deduce the formula  $\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \pi \int_0^1 \frac{1}{1+x^2} dx$

**Solution:** (a) Let  $I = \int_0^\pi xf(\sin x)dx = \int_0^\pi (\pi - x)f(\sin(\pi - x))dx = \int_0^\pi (\pi - x)f(\sin x)dx$

$$I = \pi \int_0^\pi f(\sin x)dx - \int_0^\pi xf(\sin x)dx = \pi \int_0^\pi f(\sin x)dx - I$$

$$\Rightarrow 2I = \pi \int_0^\pi f(\sin x)dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^\pi f(\sin x)dx$$

$$(b) \int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \int_0^\pi \frac{x \sin x}{2-\sin^2 x} dx = \int_0^\pi xf(\sin x)dx \text{ where } f(\sin x) = \frac{\sin x}{2-\sin^2 x}$$

$$= \frac{\pi}{2} \int_0^\pi f(\sin x)dx \text{ (using (a) )}$$

$$= \frac{\pi}{2} 2 \int_0^{\frac{\pi}{2}} f(\sin x)dx = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\sin^2 x} dx = \pi \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \frac{\sin(\frac{\pi}{2}-x)}{1+\cos^2(\frac{\pi}{2}-x)} dx = \pi \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x} dx$$

$$= \pi \int_0^1 \frac{1}{1+t^2} dt \text{ (put } t = \sin x, dt = \cos x dx\text{)}$$

$$= \pi \int_0^1 \frac{1}{1+x^2} dx \text{ (changing the dummy variable } t \text{ to } x\text{)}$$