

Yahoo Answer dated 31-08-2013

Question: In a ΔABC if $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0$ then prove that ΔABC is an isosceles triangle.

Solution: $\begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0$ implies $\begin{vmatrix} 1 & 1 & 1 \\ \sin A & \sin B & \sin C \\ \sin^2 A & \sin^2 B & \sin^2 C \end{vmatrix} = 0$ (using elementary row operations).

Applying elementary column operations, we get $\begin{vmatrix} 1 & 0 & 0 \\ \sin A & \sin B - \sin A & \sin C - \sin A \\ \sin^2 A & \sin^2 B - \sin^2 A & \sin^2 C - \sin^2 A \end{vmatrix} = 0$

$$\implies (\sin B - \sin A)(\sin C - \sin A) \begin{vmatrix} 1 & 0 & 0 \\ \sin A & 1 & 1 \\ \sin^2 A & \sin B + \sin A & \sin C + \sin A \end{vmatrix} = 0$$

$$\implies (\sin B - \sin A)(\sin C - \sin A)(\sin C - \sin B) = 0$$

$$\implies (\sin B - \sin A) = 0 \text{ or } (\sin C - \sin A) = 0 \text{ or } (\sin C - \sin B) = 0$$

Thus $B = A$ or $C = A$ or $C = B$.

Hence ΔABC is an isosceles triangle.