Seventeenth International Olympiad, 1975

1975/1.

Let x_i, y_i $(i = 1, 2, ..., n)$ be real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \text{ and } y_1 \geq y_2 \geq \cdots \geq y_n.$$

Prove that, if $z_1, z_2, \cdots, z_n$ is any permutation of $y_1, y_2, \cdots, y_n$, then

$$\sum_{i=1}^{n}(x_i - y_i)^2 \leq \sum_{i=1}^{n}(x_i - z_i)^2.$$

1975/2.

Let $a_1, a_2, a_{3,} \cdots$ be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x, y positive integers and $q > p$.

1975/3.

On the sides of an arbitrary triangle ABC, triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45°, \angle BCP = \angle ACQ = 30°, \angle ABR = \angle BAR = 15°$. Prove that $\angle QRP = 90°$ and $QR = RP$.

1975/4.

When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B. (A and B are written in decimal notation.)

1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6.

Find all polynomials P, in two variables, with the following properties:
(i) for a positive integer n and all real t, x, y

$$P(tx, ty) = t^n P(x, y)$$

(that is, P is homogeneous of degree n),
(ii) for all real a, b, c,

$$P(b+c, a) + P(c+a, b) + P(a+b, c) = 0,$$

(iii) $P(1, 0) = 1$.